Reflection of a Pyramidally Tapered Rectangular Waveguide*

KATSU MATSUMARU†

Summary—**The reflection coefficient Γ of a pyramidally tapered rectangular waveguide is derived by assuming that the taper impedance is proportional to the height and guide wavelength and inversely proportional to the width of the taper cross section. It is shown that the loci of Γ, plotted in the *K* plane as a function of taper length for some conventional tapers, do not pass through the center of the chart at multiples of a half-guide wavelength as for an exponential line, but instead they converge almost concentrically. The frequency characteristic of the pyramidally tapered waveguide is compared with other types of tapers. Typical 7-kmc experimental results for several tapers differing in length are presented.**

Introduction

In microwave systems, a pyramidal taper is often needed to connect rectangular waveguides whose ratios of width to height are equal for both the input and output terminals. The reflection coefficients of these tapers are smaller than that of either *E*- or *H*-plane tapers since the rate of change of impedance is much smaller. In this paper the reflection coefficients of these tapers are derived and general design methods are presented. Following an approximate theoretical calculation of the reflection coefficients of pyramidally tapered waveguides, two sets of loci of Γ of conventional tapers were plotted in the *K* plane to show typical performances. To confirm the formulas, a set of tapers was made to connect WR-229 and WR-159 waveguides. The calculated locus of Γ of these tapers agreed very well with the measured locus at 7.05 kmc. By using these formulas, it is fairly easy to design properly pyramidally tapered waveguides.

Calculation of Reflection Coefficients of Pyramidally Tapered Waveguide

As stated in Schelkunoff's text, the reflection coefficient of pyramidally tapered rectangular waveguide for the dominant mode is quite small compared to those of *E*- or *H*-plane tapers.[1] First, the formula of reflection coefficient of these tapered waveguides will be derived. In Fig. 1 are shown the cross sections of a tapered waveguide and the coordinate system. The width and the height of the large waveguide are a and b, respectively, and the length of the taper is h. The longitudinal length of the taper to the projected vertex is z_0. The acute angles at the vertex in the xz and yz planes are $2\theta_a$ and $2\theta_b$, respectively, as shown in Fig. 1. It is assumed that the electromagnetic wave propagates from left to right, and the dielectric constant of the medium within the waveguide is unity. The intrinsic impedance K_z of a rectangular waveguide is expressed by the following:[2]

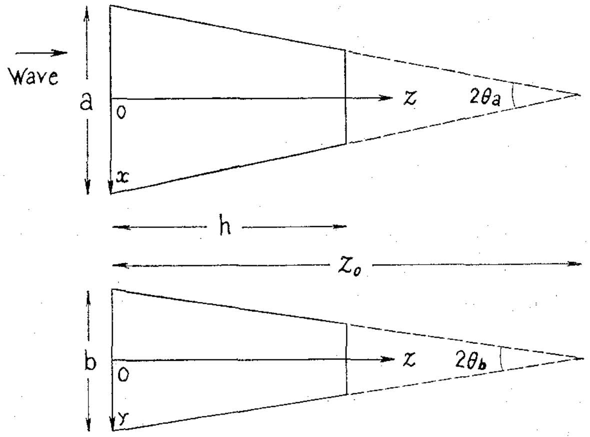

Fig. 1—Illustration of the cross sections and the coordinate system in a taper.

$$K_z = \frac{\eta}{\sqrt{1-(\lambda_0/\lambda_c(z))^2}}, \tag{1}$$

where η is a constant, λ_0 is the free space wavelength, and $\lambda_c(z)$ is the cutoff wavelength at a sectional plane $z=z$. By substituting the following relations

$$\frac{\lambda_c(z)}{2a} \equiv \frac{z_0 - z}{z_0}, \quad \text{and} \quad P = \left(\frac{z_0\lambda_0}{2a}\right)^2 \tag{2}$$

in (1), K_z can be expressed as a function of z:

$$K_z = \frac{\eta}{\sqrt{1-P/(z_0-z)^2}}. \tag{3}$$

Following Schelkunoff's definition, the integrated impedance K_{wv} of a rectangular waveguide is assumed to be proportional to its height and inversely proportional to its width.[3] By using (3), K_{wv} can be expressed as a function of z as follows:

$$K_{wv} = \frac{2\eta}{\sqrt{1-P/(z_0-z)^2}} \cdot \frac{b - 2z\tan\theta_b}{a - 2z\tan\theta_a}. \tag{4}$$

* Manuscript received by the PGMTT, March 18, 1958; revised manuscript received, December 23, 1958.

† Elec. Communication Lab., Nippon Telegraph and Telephone Public Corp., Tokyo, Japan.

[1] S. A. Schelkunoff, "Electromagnetic Waves," D. Van Nostrand Co. Inc., New York, N. Y., pp. 316–320; 1943.

[2] *Ibid.*, p. 317. See (21)-(11).

[3] *Ibid.*, p. 319. See (21)-(21).

If the rate of change of K_{wv} in the longitudinal direction is small, the reflection coefficient Γ of the taper, shown in Fig. 1, is obtained by the following integration:

$$\Gamma = \frac{1}{2}\int_{z=0}^{z=h}\left(\exp - \int_0^z 2j\beta(z)dz\right) \times \frac{dK_{wv}}{K_{wv}} \cdot \quad (5)$$

The phase constant $\beta(z)$ is a function of z and is given by

$$\beta(z) = \frac{2\pi}{\lambda_g(z)} = \frac{2\pi}{\lambda_0}\sqrt{1 - P/(z_0 - z)^2}$$
$$\doteqdot \frac{2\pi}{\lambda_0}\left(1 - \frac{P}{2}\frac{1}{(z_0 - z)^2}\right).$$

By using this approximation, the exponent in (5) is given by

$$-\int_0^z 2j\beta(z)dz \doteqdot -j\frac{4\pi}{\lambda_0}\left\{z - \frac{P}{2}\left(\frac{1}{z_0 - z} - \frac{1}{z_0}\right)\right\}. \quad (6)$$

From (4), dK_{wv}/K_{wv} can be obtained as follows:

$$\frac{dK_{wv}}{K_{wv}} = \left\{\frac{2(-a \tan\theta_b + b \tan\theta_a)}{(a - 2z\tan\theta_a)(b - 2z\tan\theta_b)} + \frac{P}{(z_0 - z)^3 - P(z_0 - z)}\right\} dz. \quad (7)$$

Since the ratios a/b are equal for both the input and output terminal waveguides,

$$-a \tan\theta_b + b \tan\theta_a = 0. \quad (8)$$

Finally, by using relations (7) and (8), the formula of Γ is

$$\Gamma = \frac{P}{2}\int_0^h \exp\left[-j\frac{4\pi}{\lambda_0}\left\{z - \frac{P}{2}\left(\frac{1}{z_0 - z} - \frac{1}{z_0}\right)\right\}\right] \times \frac{dz}{(z_0 - z)^3 - P(z_0 - z)} \cdot \quad (9)$$

Eq. (9) is the general formula for the reflection coefficient of a pyramidally tapered waveguide. In general, the magnitude of Γ is proportional to P and independent of the height of the waveguide. For specified dimensions of the terminal waveguides, the magnitude of Γ decreases in an oscillatory fashion with increase in longitudinal length h.

Putting

$$z = \lambda_0 z', \quad z_0 = \lambda_0 z_0', \quad P = \lambda_0^2 P', \quad h = \lambda_0 h',$$

(9) can be normalized as follows:

$$\Gamma = \frac{P'}{2}\int_0^{h'} \exp\left[-j4\pi\left\{z' - \frac{P'}{2}\left(\frac{1}{z_0' - z'} - \frac{1}{z_0'}\right)\right\}\right] \times \frac{dz'}{(z_0' - z')^3 - P'(z_0' - z')} \cdot \quad (10)$$

Since (10) depends only on h', z_0' and P', calculated curves of Γ of some specific examples can be used as universal design charts if a set of tables of the variables is associated with them.

Characteristics of Γ of Pyramidally Tapered Rectangular Waveguide

Inasmuch as tapered waveguides are used in conjunction with other components in a waveguide system, it is necessary to know not only the magnitude but the phase of the reflection coefficient Γ. By using graphical integration, calculated values of Γ from (10) have been published for standard rectangular waveguides over the frequency range of 2 kmc to 14 kmc.[4]

First, to illustrate the general characteristics of Γ of these tapers, the calculated results are mentioned of typical tapers designed to connect WR-229 and WR-159 waveguides. With regard to frequency characteristics, Γ is calculated for three frequencies of 7.05 kmc ($\lambda_0 = 4.25$ cm), 6.0 kmc ($\lambda_0 = 5.00$ cm) and 5.5 kmc ($\lambda_0 = 5.45$ cm) and the waveguide dimensions, etc., are compiled in Table I.

The tapers differing in length with parameters h, h', z_0', and θ_a are tabulated in Tables II–IV for 7.05, 6.0, and 5.5 kmc, respectively. The calculated values of Γ are plotted in Figs. 2–4 and these loci correspond to the tapers described in Tables II, III and IV, respectively. The reference plane of Γ is at $z = 0$, *i.e.*, at the terminal of the larger waveguide. Figs. 2–4 depict the typical frequency characteristic of Γ for general cases. It is to be noted that the magnitude of Γ decreases with increasing taper length and the loci do not pass through the center of the chart. Moreover, the phase angle of Γ sweeps almost one revolution if the taper length changes by one-half guide wavelength. Fig. 5 shows the VSWR's of these tapers as a function of frequency. It should be mentioned that the VSWR decreases almost linearly with length if the taper is shorter than $2\lambda_g$, but not for the longer tapers. Considering the benefit of practical applications, the loci of Γ in Figs. 2–4 were traced as values of the admittance, even if, following the usual fashion, the conventional notations $g - jb$ and $g + jb$ were deleted in the graphs. Therefore, in order to use them as values of the impedance, the locations of these loci should be rotated by 180° about the center in the charts. (As for Figs. 2–4, the loci of Γ in Figs. 6–8 too, were traced as values of the admittance. Therefore, these loci also should be rotated by 180° about the center, in order to use them as the impedance.

In another example, tapered waveguides, designed to connect WR-229 with WR-187 waveguides, are discussed. Table V shows the waveguide dimensions and λ_g in both terminal waveguides for three different frequencies in the 4-kmc band. (In this example, the tapers

[4] K. Matsumaru, "Considerations on tapered waveguides," *Elec. Communication Lab. Tech. J.*, vol. 7, pp. 52–62; May, 1958.

TABLE I

	Waveguide	a (in)	b (in)	λ_g (cm)		
				7.05 kmc	6.0 kmc	5.5 kmc
Input terminal	WR-229	2.290	1.145	4.6	5.5	6.5
Output terminal	WR-159	1.590	0.795	5.0	6.3	7.4

TABLE II

$f=7.05$ KMC

h (cm)	2	3	4	5	6	7
h'	0.47	0.70	0.94	1.18	1.41	1.64
z_0'	1.51	2.26	3.01	3.77	4.52	5.27
θ_a	24.3°	16.8°	12.7°	10.3°	8.6°	7.4°

TABLE III

$f=6.0$ KMC

h (cm)	2.5	4	5	6	7.5	9	10
h'	0.50	0.80	1.00	1.20	1.50	1.80	2.00
z_0'	1.60	2.57	3.21	3.85	4.80	5.75	6.40
θ_a	19.9°	12.7°	10.3°	8.6°	6.9°	5.7°	5.2°

TABLE IV

$f=5.5$ KMC

h (cm)	3	4	5	6	7	8	9	10
h'	0.55	0.73	0.92	1.10	1.28	1.47	1.65	1.83
z_0'	1.77	2.36	2.95	3.54	4.12	4.71	5.31	5.89
θ_a	16.8°	12.7°	10.3°	8.6°	7.4°	6.5°	5.7°	5.2°

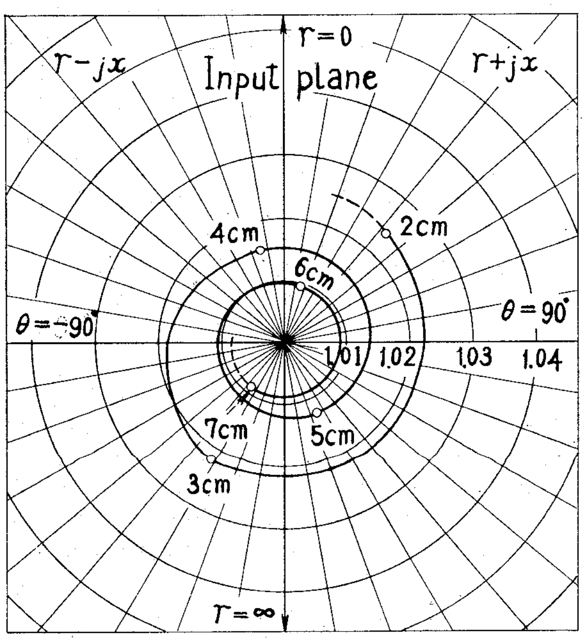

Fig. 2—The curve shows a calculated locus of Γ of typical tapers used to connect WR-229 and WR-159 waveguides. Lengths of tapers are from 2 cm to 7 cm, and the frequency is 7.05 kmc. See Tables I and II. In Figs. 2–4, loci were shown as the admittance even if the conventional notations $g-jb$ and $g+jb$ were deleted. Therefore, to use these loci as values of the impedance, they should be rotated by 180° about the centers in the diagrams.

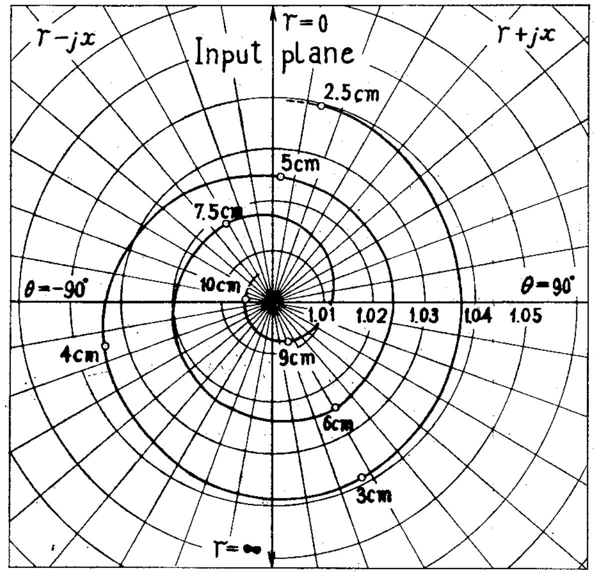

Fig. 3—The curve shows a locus of Γ of the same tapers in Fig. 2 but at a frequency of 6.0 kmc. Lengths of tapers are from 2.5 cm to 10 cm. See Tables I and III.

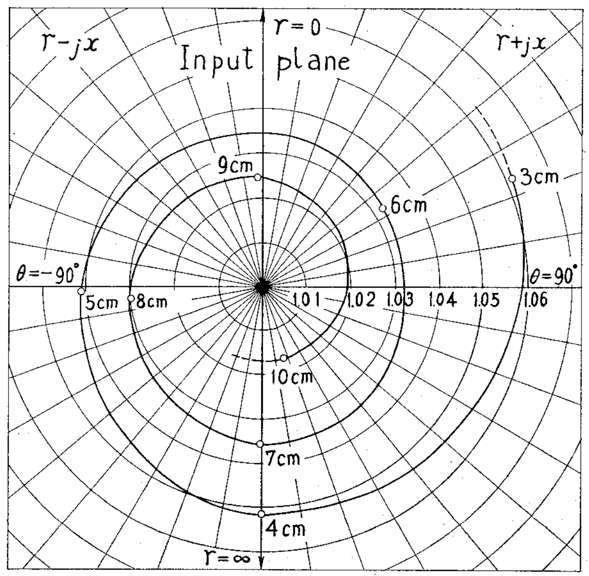

Fig. 4—The curve shows a locus of Γ of the same tapers in Fig. 2 but at a frequency of 5.5 kmc. Lengths of tapers are from 3 cm to 10 cm. See Tables I and IV.

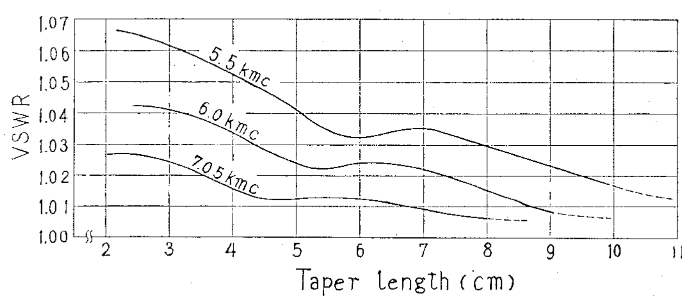

Fig. 5—The curves show the VSWR's given in Figs. 2–4 as a function of taper length for three different frequencies.

TABLE V

	Waveguide	a (in)	b (in)	λ_g (cm)		
				4.6 kmc	4.2 kmc	3.9 kmc
Input terminal	WR-229	2.290	1.145	7.9	9.0	10.3
Output terminal	WR-187	1.872	0.872	8.9	10.8	13.1

TABLE VI
f=4.6 KMC

h (cm)	7	9	11	13	15	17
h'	1.07	1.38	1.69	1.99	2.30	2.61
z_0'	5.85	7.52	9.21	10.88	12.54	14.20
θ_a	4.3°	3.4°	2.8°	2.4°	2.0°	1.8°

TABLE VII
f=4.2 KMC

h (cm)	7	9	11	13	15	17
h'	0.98	1.26	1.54	1.82	2.10	2.38
z_0'	5.35	6.86	8.40	9.94	11.45	12.99
θ_a	4.3°	3.4°	2.8°	2.4°	2.0°	1.8°

TABLE VIII
f=3.9 KMC

h (cm)	7	9	11	13	15	17
h'	0.91	1.17	1.43	1.69	1.95	2.21
z_0'	4.96	6.38	7.82	9.20	10.62	12.01
θ_a	4.3°	3.4°	2.8°	2.4°	2.0°	1.8°

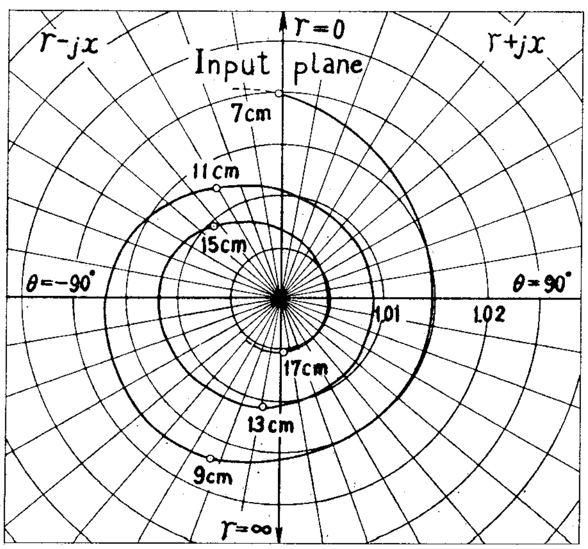

Fig. 6—The curve shows a calculated locus of Γ of typical tapers used to connect WR-229 and WR-187 waveguides. Lengths of tapers are from 7 cm to 17 cm, and the frequency is 4.6 kmc. See Tables V and VI. In Figs. 6–8, loci were shown as the admittance in the same way as those in Figs. 2–4. Therefore, to use these loci as values of the impedance, they also should be rotated by 180° about the centers.

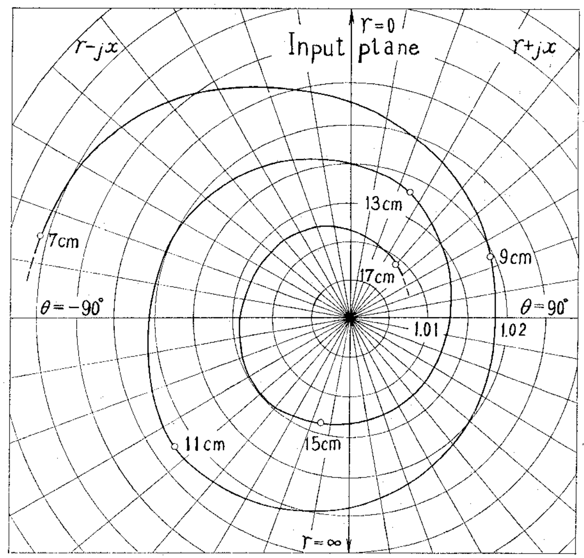

Fig. 7—The curve shows a locus of Γ of the same tapers in Fig. 6 but at a frequency of 4.2 kmc. See Tables V and VII.

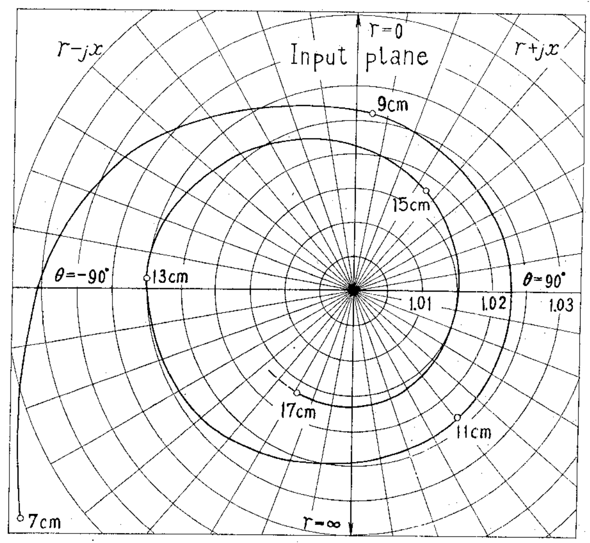

Fig. 8—The curve shows a locus of Γ of the same tapers in Fig. 6 but at a frequency of 3.9 kmc. See Tables V and VIII.

are not exactly pyramidal, *i.e.*, $a_1/b_1 \neq a_2/b_2$.) Tables VI–VIII list taper parameters h, h', z_0', and θ_a for 4.6, 4.2, and 3.9 kmc, respectively. Figs. 6–8 show the loci of Γ calculated by an approximate method of graphical integration for the tapers listed in Tables VI–VIII. By using the loci of Γ as illustrated in the examples given in Figs. 2–8, it is possible to design any pyramidally tapered waveguide with considerable accuracy. If Γ is calculated from (10) for two or three discrete taper lengths, a spiral locus of Γ can be traced through the two or three points as illustrated in the above examples.

Finally the behavior of reflection characteristics of these tapers are compared with other tapers. For exponential tapers, Ragan shows a typical figure of VSWR's which reduces to unity at multiples of a half wavelength and humps regularly.[5] However, as seen from Fig. 5, the VSWR's of these tapers do not reduce to unity for particular taper lengths. Next the reflection characteristics of linear tapers, discussed recently,[6] are somewhat similar to those of the subject pyramidal tapers. However, the reflection of an E-plane linear taper is more critically dependent on waveguide dimension (height of waveguide) rather than on frequency. For these E-plane linear tapers, the frequency behavior of Γ is quite similar to that shown in Fig. 5. From Stevenson's theoretical calculations on electromagnetic horns, (10) is valid for small flare angles.[7] Since the flare angles of the tapers are usually small, except for extremely short tapers, the values of Γ can be calculated from (10) with very little error.

Experimental Results

Inasmuch as the derivation of (10) is based on a number of assumptions, the calculated results should be checked experimentally. Measurements were made on the (WR-229/WR-159) tapers described in Tables I and II at a frequency of 7.05 kmc. Since the Γ's of these tapers are very small, they were measured as accurately as possible. The lengths of tapers varied from 3 cm to 7 cm in intervals of 1 cm, and the data were taken at a frequency of 7.05 kmc. The reflection coefficients Γ were measured in the smaller WR-159 waveguide and for this case Γ is given by

$$\Gamma=\frac{P}{2}\int_0^h \exp j\left\{\pi-\frac{4\pi}{\lambda_0}\left(z-\frac{P}{2}\left(\frac{1}{z_0-h}-\frac{1}{z_0-h+z}\right)\right)\right\} \times\frac{dz}{(z_0-h+z)^3-P(z_0-h+z)}. \quad (11)$$

Measured results are shown in Fig. 9, and these agree quite well with the data calculated from (11). The mean error in VSWR was as small as 0.004. The conically looped locus of Γ is quite similar to that of a linear taper.[8] Of course, the calculated Γ shown in this graph corresponds to Fig. 2 through a simple transformation. Other measurements were made at frequencies lower than 7.05 kmc, and the agreement was worse, as expected. In the frequency range approximately covered by Figs. 2–4, the mean error in the VSWR is less than 0.02 for VSWR smaller than 1.08. As seen from Figs. 2–8, the VSWR's of these particular tapers are smaller than 1.07 or 1.08, and therefore, in practice, the VSWR's computed from (10) or (11) should agree with observed values to within 0.02. For tapers whose VSWR's are smaller than 1.05, the mean error will be as small as 0.01.

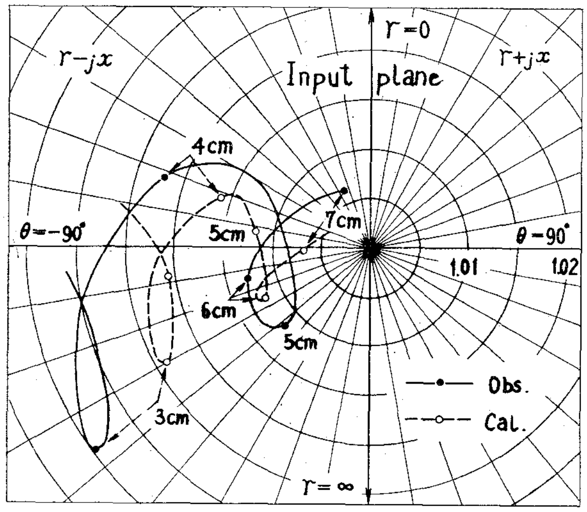

Fig. 9—Measured results of Γ of pyramidally tapered waveguides. The waveguides of the input and output terminals are WR-159 and WR-229. Lengths of tapers are from 3 cm to 7 cm. The solid and dashed curves show the observed and calculated values, respectively. The frequency is 7.05 kmc.

Conclusion

The reflection coefficients of pyramidally tapered waveguides can be calculated from (10). As seen from the examples, the VSWR's of typical tapers are smaller than about 1.08, and for most cases, the calculated VSWR's should agree with the experimental values to within 0.02. As shown in Figs. 2–8, Γ decreases almost concentrically for the tapers shorter than $2\lambda_g$, and the phase angle sweeps almost one cycle every half a guide wavelength of taper length. Consequently, a spiral of Γ can be traced in the K plane by calculating Γ for two or three discrete values of taper lengths. For tapers longer than $2\lambda_g$ the VSWR does not decrease significantly with the taper length.

Acknowledgment

The author is indebted to his colleagues at the Electrical Communication Laboratory for assistance in undertaking this research. Thanks are also due to K. Tomiyasu and to the PGMTT reviewers for their suggestions on the manuscript.

[5] G. L. Ragan, "Microwave Transmission Circuit," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 9, p. 307; 1948. See Fig. 6.3.
[6] K. Matsumaru, "Reflection coefficient of E-plane tapered waveguides," IRE Trans. on Microwave Theory and Techniques, vol. MTT-6, pp. 143–149; April, 1958.
[7] A. F. Stevenson, "General theory of electromagnetic horns," *J. Appl. Phys.*, vol. 22, pp. 1447–1460; December, 1951.
[8] K. Matsumaru, "Rebuttal to R. F. H. Yang's comments," IRE Trans. on Microwave Theory and Techniques, vol. MTT-7, pp. 175–176; January, 1959.